

## Complex Functions

$$z^n = r^n (\cos(n\theta) + j \sin(n\theta)) = r^n e^{nj\theta}$$

$$\theta = \arctan\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right) \quad r = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2}$$

$$\sqrt[n]{z} = \sqrt[n]{r} \left( \cos\left(\frac{\theta+2k\pi}{n}\right) + j \sin\left(\frac{\theta+2k\pi}{n}\right) \right), k = 1..n-1$$

$$\cos(z) = \frac{e^{jz} + e^{-jz}}{2} \quad \sin(z) = \frac{e^{jz} - e^{-jz}}{2j}$$

$$\cosh(z) = \frac{e^z + e^{-z}}{2} \quad \sinh(z) = \frac{e^z - e^{-z}}{2}$$

$$\cosh(jz) = \cos(z) \quad \sinh(jz) = j \sin(z)$$

$$\cosh(z) = \cos(jz) \quad \sinh(z) = j \sin(jz)$$

$$\ln(z) = \ln(r) + j\theta$$

$$Ln(z) = \ln|z| + jArg(z)$$

$$\ln(z) = Ln(z) \pm 2n\pi j$$

## Cauchy Riemann Test

Analytic where:  $f(z) = u(x, y) + jv(x, y)$

$$\frac{du}{dx} = \frac{dv}{dy} \quad \text{and} \quad \frac{du}{dy} = -\frac{dv}{dx}$$

or when:  $f(z) = u(r, \theta) + jv(r, \theta)$

$$\frac{du}{dr} = \frac{1}{r} \frac{dv}{d\theta} \quad \text{and} \quad \frac{du}{dr} = -\frac{1}{r} \frac{dv}{d\theta}$$

must also be single valued.

$$f'(z) = \frac{du}{dx} + j \frac{dv}{dx} \quad f'(z) = \frac{du}{dy} - j \frac{dv}{dy}$$

Analytic Functions are harmonic

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

## Complex Integrals

In general:

$$\int_c f(z) dz = \int_a^b f[z(t)] \dot{z}(t) dt$$

If analytic in a simply connected domain:

$$\int_a^b f(z) dz = F(b) - F(a) \quad \text{where} \quad F'(z) = f(z)$$

$$\oint_c f(z) dz = 0$$

When enclosing poles:

$$\oint_c \frac{f(z)}{z-z_0} dz = 2\pi j f(z_0)$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi j} \oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$\oint_c f(z) dz = 2\pi j \sum_{k=0}^n \text{Res}(f(z))$$

$$\text{Res}_{z=z_0} f(z) = \lim_{z \rightarrow z_0} (z-z_0) f(z) \quad \text{first order}$$

$$\text{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] \right\}$$

$$\text{Res}_{z=z_0} f(z) = \text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p'(z_0)}{q'(z_0)}$$

## Series Expansions

### Laurent Series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad a_n = \frac{1}{2\pi j} \oint_c \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*$$

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad g(z) = \sum_{k=0}^{\infty} b_k z^k$$

$$f(z)g(z) = \sum_{k=0}^{\infty} (a_0 b_n + \dots + a_n b_0) z^k$$

### Various Common Series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad Ln(z+1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n!}$$

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad \sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad \sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

## Laplace Transforms

$f(t)$	$\mathcal{L}(f(t))$
1	$\frac{1}{s}$
$t^n$	$\frac{n!}{s^{n+1}}$
$t^a$	$\frac{\Gamma(a+1)}{s^{a+1}}$
$e^{at}$	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh(at)$	$\frac{s}{s^2 - a^2}$
$\sinh(\omega t)$	$\frac{a}{s^2 - a^2}$
$f'$	$sF(s) - f(0)$
$f''$	$s^2 F(s) - sf(0) - f'(0)$
$f^{(n)}$	$s^n F(s) - \sum_{m=0}^{n-1} s^{n-m} f^{(m-1)}(0)$
$\int_0^t f(t) dt$	$\frac{1}{s} F(s)$
$e^{at} f(t)$	$F(s-a)$
$\delta(t-a)$	$e^{-as}$
$f(t-a)u(t-a)$	$e^{-as} F(s)$

## Partial Fraction Expansion

### Unrepeated Factors $(s-a)$

$$A_n = \text{Res}_{z=z_n} F(s)$$

### Repeated Factors $(s-a)^m$

$$A_k = \frac{1}{(m-k)!} \lim_{s \rightarrow a} \left\{ \frac{d^{m-k}}{ds^{m-k}} [(s-a)^m F(s)] \right\}$$